PRICING FORMULA FOR EXCHANGE OPTION IN FRACTIONAL BLACK-SCHOLES MODEL WITH JUMPS

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Abstract. In this paper pricing formula for exchange option in a fractional Black-Scholes model with jumps is derived. We found out some errors in proof of pricing formula for European call option [7]. At first we revise these errors and then extend this result to pricing formula for exchange option in fractional Black-Scholes model with jumps.

Key Words: Pricing formula, Exchange option, Fractional Black-Scholes model, Jump noise.

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1. Introduction

Fractional Black-Scholes model with jumps is as follows [7].

\begin{equation}
\begin{aligned}
\, dB(t) &= (r_d - r_f)B(t)dt, \quad B(0) = 1, \\
\, dS(t) &= S(t) \left( (\mu - \lambda \xi)dt + \sigma dB_H(t) + (e^\xi - 1)dN(t) \right), \\
\, S(0) &= S,
\end{aligned}
\end{equation}

(1.1)

where \( r_d, r_f \) are the short-term domestic interest rate and foreign interest rate respectively, and these are known. \( S(t) \) denotes the spot exchange rate at time \( t \) and \( \mu, \sigma \) are assumed to be constants. \( B_H(t) \) is a fractional Brownian motion and \( N(t) \) is a Poisson process with rate \( \lambda \). \( \xi(t) \) is jump size percent at time \( t \) which is sequence of independent
identically distributed, and \((e^{\xi(t)} - 1) \sim N\left(\mu_{\xi(t)}, \sigma_{\xi(t)}^2\right)\). In addition, all three sources of randomness, the fractional Brownian motion \(B_H(t)\), the Poisson process \(N(t)\) and jump size \(e^{\xi(t)} - 1\) are assumed to be independent.

Currencies are different with stocks; moreover since geometric Brownian motion cannot represent movement currency returns precisely, some papers have provided evidence of mispricing for currency options by standard option price model [1]. Merton proposed a jump-diffusion process with Poisson jump to match the abnormal fluctuation of stock price [3, 5]. Non-normality, non-independence and nonlinearity were discovered in empirical researches of currency return processes. To capture these non-normal behaviors, scholars have considered other distributions with fat tails such as Pareto-stable distribution and tried to interpret long memory and self-similarity using fractional Brownian motion. Research interest for interpreting these abnormal phenomena was re-encouraged by new insights in stochastic analysis based on the Wick integration [2]. Neucula and Meng et al. derived fractional Black-Scholes formula for option pricing using geometric fractional Brownian motion [6, 4]. Model (1.1), the combination of Poisson jumps and fractional Brownian motion was introduced and pricing formula for European call option was derived in [7], but we found out some errors in evaluation of quasi-expectation. In this paper we revise pricing formula for European call option and derive pricing formula for exchange option in fractional Black-Scholes model with jumps and so generalize previous pricing formula for European call option.

2. Preliminaries

We describe some necessary lemmas.

**Lemma 2.1.** ([6]) (Geometric fractional Brownian motion) Consider the fractional differential equation

\[
dX(t) = X(t) \left(\mu dt + \sigma dB_H(t)\right), \quad X(0) = x.
\]

We have that

\[
X(t) = x \exp \left(\sigma B_H(t) + \mu t - \frac{1}{2} \sigma^2 t^{2H}\right).
\]
Pricing formula for exchange option in fractional Black-Scholes model with jumps

Lemma 2.2. ([6]) Let $f$ be a function such that $E[f(B_H(T))]<\infty$. Then for every $t<T$ we have

$$
\tilde{E}_t[f(B_H(T))] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(T^2H-t^2H)}} \exp\left(-\frac{(x-B_H(t))^2}{2(T^2H-t^2H)}\right) f(x) dx,
$$

where $\tilde{E}[-]$ denotes quasi-conditional expectation with respect to $F_t^H = B(H(s), s < t)$. That is for $G = \sum_{n=0}^{\infty} \int_{\mathbb{R}} g_n dB^{\otimes n}_H \in G^*$ we define as

$$
\tilde{E}_t[G] := \tilde{E}[G|F_t^H] = \sum_{n=0}^{\infty} \int_{\mathbb{R}} g_n(s) \chi_{0 \leq s \leq t}(s) dB^{\otimes n}_H(s).
$$

Let $\theta \in \mathbb{R}$. Consider the process

$$
B_H^\theta(t) = B_H(t) + \theta t^{2H} = B_H(t) + \int_0^t 2H\theta r^{2H-1} dr,
$$

This process is a fractional Brownian motion under new measure $\mu^*$ by fractional Girsanov theorem, where measure $\mu^*$ is defined as $d\mu^*/d\mu = Z(t) = \exp\left(-\theta B_H(t) - \frac{\theta^2}{2} t^{2H}\right)$. We will denote by $\tilde{E}_t^*[-]$ the quasi-conditional expectation with respect to $\mu^*$.

Lemma 2.3. ([6]) Let $f$ be a function such that $E[f(B_H(T))]<\infty$. Then for every $t<T$

$$
\tilde{E}_t^*[f(B_H(T))] = \frac{1}{Z(t)} \tilde{E}_t[f(B_H(T))Z(T)].
$$

Lemma 2.4. ([6]) (fractional risk-neutral evaluation) The price at every $t \in [0, T]$ of a bounded $F_t^H$-measurable claim $F \in L^2(\mu)$ is given by $F(t) = e^{-r(T-t)} \tilde{E}_t[F]$.

3. Main results

Theorem 3.1. In fractional Black-Scholes model (1.1) with jumps, pricing formula for European call option is as follows.

$$
V(S(t), t) = \sum_{n=0}^{\infty} \frac{\lambda^n(T-t)^n}{n!} e^{-\lambda(T-t)} S_n
$$

$$
\times \left\{ S(t) \exp\left(-\lambda \mu \xi (T-t) + \sum_{j=1}^{n} \xi_j \right) \Phi(d_+) - Ke^{-(r_d-r_f)(T-t)} \Phi(d_-) \right\},
$$
where $\varepsilon_n$ denotes the expectation operator over the distribution of $\exp\left(\sum_{j=1}^n \xi_j\right)$ and

$$d_\pm = \frac{\ln(S(t)/K) + \sum_{j=1}^n \xi_j + (r_d - r_f - \lambda\mu\xi)(T - t)}{\sigma\sqrt{T^{2H} - t^{2H}}} \pm \frac{1}{2}\sigma\sqrt{T^{2H} - t^{2H}}.$$

**Proof.** It was proved in [2] that model (1.1) is complete and does not have an arbitrage opportunity. Thus under risk-neutral measure $\hat{P}_H$ model (1.1) can be expressed as

$$dS(t) = S(t) \left\{ (r_d - r_f)dt + \sigma d\hat{B}_H(t) + (e^\xi - 1)dN(t) \right\},$$

where risk-neutral measure $\hat{P}_H$ is defined as

$$\frac{d\hat{P}_H}{dP_H} = \exp\left\{-\theta B_H(t) - \frac{\theta^2}{2}t^{2H}\right\},$$

under this measure process $\hat{B}_H(t) = B_H(t) + \theta t^{2H}$ is a fractional Brownian motion and $\theta = (\mu - \lambda\mu\xi + r_f - r_d)/\sigma$. By [7] the solution of Eq. (3.1) is expressed as

$$S(T) = S(t)\exp\left\{ (r_d - r_f - \lambda\mu\xi)(T - t) - \frac{1}{2}\sigma^2(T^{2H} - t^{2H}) \right.$$  

$$+\sigma(\hat{B}_H(T) - \hat{B}_H(t)) + \sum_{j=1}^{N(T-t)} \xi_j \right\},$$

By Lemma 2.4 the price at $t$ for European call option $F = (S(T) - K)^+$ is

$$V(S(t), t) = e^{-(r_d - r_f)(T - t)}\tilde{E}_t[F] = e^{-(r_d - r_f)(T - t)}\tilde{E}_{\hat{P}_H}[F|\mathcal{F}_t].$$

If we define as

$$S_n(T) = S(t)\exp\left\{ (r_d - r_f - \lambda\mu\xi)(T - t) - \frac{1}{2}\sigma^2(T^{2H} - t^{2H}) \right.$$  

$$+\sigma(\hat{B}_H(T) - \hat{B}_H(t)) + \sum_{j=1}^n \xi_j \right\},$$
then
\[ V(S(t), t) = e^{-(r_d - r_f)(T-t)} \tilde{E}_{\tilde{P}_H}[(S(T) - K)^+ | F_t^H] \]
\[ = e^{-(r_d - r_f)(T-t)} \sum_{n=0}^{\infty} \frac{\lambda^n(T-t)^n}{n!} e^{-\lambda(T-t)} \tilde{E}_{\tilde{P}_H} \]
\[ [(S_n(T) - K)^+ | F_t^H]. \]

(3.2)

Since
\[ \tilde{E}_{\tilde{P}_H}[(S_n(T) - K)^+ | F_t^H] = \tilde{E}_{\tilde{P}_H} [S_n(T) \chi_{\{S_n(T) > K\}} | F_t^H] \]
\[ - K \tilde{E}_{\tilde{P}_H} [\chi_{\{S_n(T) > K\}} | F_t^H], \]

we firstly estimate \( \tilde{E}_{\tilde{P}_H} [\chi_{\{S_n(T) > K\}} | F_t^H]. \) From Lemma 2.2, we have
\[ \tilde{E}_{\tilde{P}_H} [\chi_{\{S_n(T) > K\}} | F_t^H] = \tilde{E}_{\tilde{P}_H} [\chi_{\{B_H(t) > d_+\}} | F_t^H] \]
\[ = \frac{1}{\sqrt{2\pi(T^{2H} - t^{2H})}} \int_{d_+}^{\infty} \exp \left( -\frac{(x - \hat{B}_H(t))^2}{2(T^{2H} - t^{2H})} \right) dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{d_+ - \hat{B}_H(t)}^{\infty} \exp \left( -\frac{y^2}{2} \right) dy \]
\[ = \Phi \left( \frac{\hat{B}_H(t) - d_+}{\sqrt{T^{2H} - t^{2H}}} \right) \]
\[ = \Phi(d_-), \]

(3.4)

where
\[ d_- = \left( \ln(K/S(t)) - (r_d - r_f - \lambda \mu) (T-t) + \frac{1}{2} \sigma^2 (T^{2H} - t^{2H}) \right) \]
\[ - \sum_{j=1}^{n} \xi_j + \sigma \hat{B}_H(t) / \sigma. \]

Next we estimate \( \tilde{E}_{\tilde{P}_H} [S_n(T) \chi_{\{S_n(T) > K\}} | F_t^H]. \) Let \( B_H^*(t) = \hat{B}_H(t) - \sigma t^{2H}, \) then from fractional Girsanov formula, there exists a probability measure \( P_H^* \) such that \( B_H^*(t) \) is a fractional Brownian motion. In fact, the probability measure \( P_H^* \) is defined as follows:
\[ \frac{dP_H^*}{dP} = \exp \left\{ \sigma d\hat{B}_H(t) - \frac{1}{2} \sigma^2 t^{2H} \right\} = Z(t). \]
From Lemma 2.3 we have
\[
\tilde{E}_{P_H}[S_n(T)\chi_{\{S_n(T) > K\}}|\mathcal{F}_t^H]
\]
\[
= S \exp \left( (r_d - r_f - \lambda \mu \xi) T + \sum_{j=1}^{n} \xi_j \right)
\times \tilde{E}_{P_H}[Z(T)\chi_{\{S_n(T) > K\}}|\mathcal{F}_t^H]
\]
\[
= S \exp \left( (r_d - r_f - \lambda \mu \xi) T + \sum_{j=1}^{n} \xi_j \right)
\times Z(t) \tilde{E}_{P_H}[\chi_{\{B_H(T) > d_+\}}|\mathcal{F}_t^H]
\]
\[
= S(t) \exp \left( (r_d - r_f - \lambda \mu \xi) (T - t) + \sum_{j=1}^{n} \xi_j \right)
\times \tilde{E}_{P_H}[\chi_{\{B_H(T) > d_+\}}|\mathcal{F}_t^H]
\]
\[
= S(t) \exp \left( (r_d - r_f - \lambda \mu \xi) (T - t) + \sum_{j=1}^{n} \xi_j \right) \Phi(d_+),
\]
where
\[
d_+^* = d_-^* - \sigma T^{2H},
\]
\[
d_+ = \frac{B_H^*(t) - d_+^*}{\sqrt{T^{2H} - t^{2H}}} = d_- + \sigma (T^{2H} - t^{2H}).
\]

Now substituting Eq. (3.4) and Eq. (3.5) into Eq. (3.2) and Eq. (3.3) implies the statement of the theorem. \(\square\)

**Theorem 3.2.** In fractional Black-Scholes model (1.1) with jump noise, pricing formula for exchange option of two foreign currencies is as follows.

\[
V(S(t), t) = \sum_{n=0}^{\infty} \frac{\lambda^n (T - t)^n}{n!} e^{-\lambda(T-t)} \left\{ S_1(t) \exp \left( -\lambda \mu \xi (T - t) + \sum_{j=1}^{n} \xi_j^{(1)} \right) \Phi(\tilde{d}_+) - S_2(t) \exp \left( -\lambda \mu \xi (T - t) + \sum_{j=1}^{n} \xi_j^{(2)} \right) \Phi(\tilde{d}_-) \right\},
\]
where $\tilde{d}_\pm$ denotes the expectation operator over the distribution of $\exp\left(\sum_{j=1}^n \xi_j\right)$ and

$$
\tilde{d}_\pm = \ln\left(\frac{S_1(t)}{S_2(t)}\right) + \frac{1}{2}(\sigma_1 - \sigma_2)^2(T^{2H} - t^{2H}) + \sum_{j=1}^n \left(\xi_j^{(1)} - \xi_j^{(2)}\right).
$$

Proof. Under the risk-neutral measure $\hat{P}_H$, Exchange rates for two foreign currencies $S_1(t), S_2(t)$ satisfy the following equations:

$$
dS_1(t) = S_1(t) \left\{ (r_d - r_f)dt + \sigma_1 d\hat{B}_H(t) + (e^{\xi_1} - 1)dN(t) \right\},
$$

$$
dS_2(t) = S_2(t) \left\{ (r_d - r_f)dt + \sigma_2 d\hat{B}_H(t) + (e^{\xi_2} - 1)dN(t) \right\}.
$$

Using Lemma 2.4, we have the price of exchange option at $t$

$$
V(S_1(t), S_2(t), t) = e^{-(r_d-r_f)(T-t)} \tilde{E}_{\hat{P}_H}[ (S_1(T) - S_2(T))^+ \mid \mathbb{F}_t^H ]
$$

$$
= e^{-(r_d-r_f)(T-t)} \sum_{n=0}^{\infty} \frac{\lambda^n(T-t)^n}{n!} e^{-\lambda(T-t)} \tilde{E}_{\hat{P}_H}
$$

$$
\times [(S_1^n(T) - S_2^n(T))^+ \mid \mathbb{F}_t^H],
$$

where

$$
S_1^n(T) = S_1(t)\exp\left\{ (r_d - r_f - \lambda\mu_\xi)(T-t) - \frac{1}{2}\sigma_i^2(T^{2H} - t^{2H}) \right. 
$$

$$
+ \sigma_i(\hat{B}_H(T) - \hat{B}_H(t)) + \sum_{j=1}^n \xi_j^{(i)} \bigg\}.
$$

Also we see that the following facts hold:

$$
\tilde{E}_{\hat{P}_H}[ (S_1^n(T) - S_2^n(T))^+ \mid \mathbb{F}_t^H] = \tilde{E}_{\hat{P}_H} \left[ S_2^n(T) \left( \frac{S_1^n(T)}{S_2^n(T)} - 1 \right)^+ \right] \mid \mathbb{F}_t^H. \right.
$$

Now let

$$
\frac{dQ_H}{d\hat{P}_H} = \exp \left\{ \sigma_2 \hat{B}_H(t) - \frac{1}{2}\sigma_2^2 t^{2H} \right\} = \tilde{Z}(t).
$$
Then under this measure $Q_H$, $\tilde{B}_H(t) = \hat{B}_H(t) - \sigma_2^2 t^{2H}$ is a fractional Brownian motion and from Lemma 2.3 we have

\[
\begin{align*}
\mathbb{E}_{P_H}\left[ S_{2n}(T) \left( \frac{S_{1n}(T)}{S_{2n}(T)} - 1 \right)^+ \left| \mathcal{F}_t^H \right. \right] \\
= \mathbb{E}_{P_H}\left[ S_2 \exp \left\{ (r_d - r_f - \lambda \mu \xi)T + \sum_{j=1}^{n} \xi_j^{(2)} \right\} \tilde{Z}(T) \right. \\
\times \left( \frac{S_{1n}(T)}{S_{2n}(T)} - 1 \right)^+ \left| \mathcal{F}_t^H \right. \right] \\
= S_2 \exp \left\{ (r_d - r_f - \lambda \mu \xi)T + \sum_{j=1}^{n} \xi_j^{(2)} \right\} \tilde{Z}(t) \mathbb{E}_{Q_H} \\
\times \left( \frac{S_{1n}(T)}{S_{2n}(T)} - 1 \right)^+ \left| \mathcal{F}_t^H \right. \right] \\
= S_2 \exp \left\{ (r_d - r_f - \lambda \mu \xi)(T-t) \right\} \mathbb{E}_{Q_H} \\
\times \left( \frac{S_{1n}(T)}{S_{2n}(T)} - 1 \right)^+ \left| \mathcal{F}_t^H \right. \right] \\
= S_2(t) \exp \left\{ (r_d - r_f - \lambda \mu \xi)(T-t) + \sum_{j=1}^{n} \xi_j^{(2)} \right\} \mathbb{E}_{Q_H} \\
\times \left( \frac{S_{1n}(T)}{S_{2n}(T)} - 1 \right)^+ \left| \mathcal{F}_t^H \right. \right].
\end{align*}
\]

(3.7)

Setting $t = 0, T = t$ and considering the expression of $S_{1n}(T)$, we have

\[
\frac{S_{1n}(t)}{S_{2n}(t)} = \frac{S_1}{S_2} \exp \left\{ (\sigma_1 - \sigma_2)d\hat{B}_H(t) - \frac{1}{2}(\sigma_1^2 - \sigma_2^2)t^{2H} + \sum_{j=1}^{n} (\xi_j^{(1)} - \xi_j^{(2)}) \right\} \\
= \frac{S_1}{S_2} \exp \left\{ (\sigma_1 - \sigma_2)d\hat{B}_H(t) - \frac{1}{2}(\sigma_1 - \sigma_2)^2 t^{2H} + \sum_{j=1}^{n} (\xi_j^{(1)} - \xi_j^{(2)}) \right\}. 
\]
Thus stochastic process \( \frac{S_{1n}(t)}{S_{2n}(t)} \) satisfies the following stochastic differential equation
\[
d\left( \frac{S_{1n}(t)}{S_{2n}(t)} \right) = \frac{S_{1n}(t)}{S_{2n}(t)} \left( (\sigma_1 - \sigma_2) d\tilde{B}_H(t) + \left( e^{\xi(1)} - e^{\xi(2)} \right) dN(t) \right),
\]
so quasi-conditional expectation in Eq. (3.7) can be considered as a price for European call option with exercise price \( K=1 \). Since this is the special case of Theorem 3.1 with the parameters
\[
S_n(T) = \frac{S_{1n}(T)}{S_{2n}(T)}, \ r_d - r_f = 0, \ \sigma = \sigma_1 - \sigma_2, \ \xi = \xi(1) - \xi(2), \ \mu_\xi = 0, \ K = 1,
\]
we have
\[
\tilde{E}_{Q_H} \left[ \left( \frac{S_{1n}(T)}{S_{2n}(T)} - 1 \right) \bigg| \mathcal{F}_t \right] = \frac{S_1(t)}{S_2(t)} \exp \left\{ \sum_{j=1}^{n} (\xi_j^{(1)} - \xi_j^{(2)}) \right\} \Phi(\tilde{d}_+) - \Phi(\tilde{d}_-).
\]
Thus substituting above equation into Eq. (3.7) and again into Eq. (3.6), we obtain the result of theorem. □

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