CODING THEORY AND HYPER BCK-ALGEBRAS

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ABSTRACT. In this paper we define the notion of a hyper BCK valued function on a set and investigate some of its related properties as Y.B. Jun, S.Z. Song and C. Flaut have done for a BCK-algebras. We construct the codes generated by a hyper BCK valued function and provide an algorithm which allow to find a hyper BCK-algebra starting from a given binary block code. Moreover we establish the link between the hyper BCK-algebra constructed from a binary block code and hyper BCK-ideal on a hyper BCK-algebra.

Key Words: Hyper BCK-algebra, Coding theory, Block code, Hyper BCK-ideal.

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1. Introduction

The hyperstructure theory (called also multialgebra) is introduced in 1934 by F. Marty [7]. Since then a great deal of literature has been produced on the applications of the hyperstructures. Later K. Iseki [4] initiated in 1966 the study of BCK-algebras as a generalization of the concept of set-theoretic difference and propositional calculi. Y.B. Jun et al. [6] applied for the first time the hyperstructures to BCK-algebra and introduced in 2000 the notion of a hyper BCK-algebra with is a generalization of BCK-algebra.

Y.B. Jun and S.Z. Song, C. Flaut and T.S. Atamewoue et al. [1, 3, 5] study the connection between BCK-algebras, residuated lattices and coding theory.
The main purpose of this paper is to study coding theory in the context of hyper BCK-algebras. This work is organized as follows: In section 2, we present some basic notions about hyper BCK-algebraic that we will use in the sequel. In section 3, we introduce the notion of hyper BCK-valued functions and investigated several of their properties. In section 4, we give the construction of the block codes by using the notion of hyper BCK-valued functions, and after haves show that in some circumstances every finite hyper BCK-algebras determines a binary block code, we end by a link between the constructed block codes and some hyper BCK-ideal.

2. Preliminaries

We will recall some known concepts related to hyper BCK-algebra which will be helpful in further section. For more about hyper BCK-algebra we refer the reader to [2, 6, 8]. Let $H$ be a non-empty set endowed with a hyperoperation "$*$", i.e. a mapping of $H \times H$ into the family of nonempty subsets of $H$. For two subsets $A$ and $B$ of $H$, denote by $A*B$ the set $\bigcup_{a \in A, b \in B} a * b$. We shall use $x*y$ instead of $x*\{y\}$, $\{x\}*y$, or $\{x\} * \{y\}$.

**Definition 2.1.** By a hyper BCK-algebra we mean a non-empty set $H$ endowed with a hyperoperation $*$ and a constant $\theta$ satisfying the following axioms for all $x,y,z \in H$:

(i) $(x * z) * (y * z) \ll x * y$,
(ii) $(x * y) * z = (x * z) * y$,
(iii) $x * y \ll \{x\}$,
(vi) $x \ll y$ and $y \ll x$ imply $x = y$,

Where $x \ll y$ is defined by $\theta \in x * y$ and $A \ll B$ by for all $a \in A$, there exists $b \in B$ such that $a \ll b$, for every $A, B \subseteq H$. Note that "$\ll$" is called hyper order in $H$.

In any hyper BCK-algebra $(H, *, \theta)$ the following hold for all $x,y,z \in H$:

(a1) $x * \theta = \{x\}$, $\theta * x = \{\theta\}$ and $\theta * \theta = \{\theta\}$,
(a2) $\theta \ll x$,
(a3) $x * \theta \ll \{y\}$ implies $x \ll y$ and $y * x \ll z * x$,
(a4) $y \ll z$ implies $x * z \ll x * y$,
(a5) $x * y = \{\theta\}$ implies $(x * z) * (y * z) = \{\theta\}$. 


Definition 2.2. Let $I$ be a non-empty subset of a hyper BCK-algebra $H$. Then $I$ is called a hyper BCK-ideal of $H$ if the following hold:
(i) $\theta \in I$,
(ii) $x * y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Definition 2.3. Let $I$ be a non-empty subset of a hyper BCK-algebra $H$. Then $I$ is called a weak hyper BCK-ideal of $H$ if the following hold:
(i) $\theta \in I$,
(ii) $(x * y) \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Remark 2.4. Every hyper BCK-ideal of a hyper BCK-algebra $H$ is a weak hyper BCK-ideal of $H$, but the converse may not be true [6].

3. Hyper BCK-valued functions

In what follows let $A$ and $H$ denote a nonempty set and a hyper BCK-algebra respectively, unless otherwise specified.

Definition 3.1. A mapping $\tilde{A} : A \to H$ is called a hyper BCK-valued function (briefly, hyper BCK-function) on $A$.

Definition 3.2. A cut function of $\tilde{A}$, for $q \in H$ is defined to be a mapping $\tilde{A}_q : A \to \{0, 1\}$ such that $(\forall x \in A) (\tilde{A}_q(x) = 1 \iff \theta \in q * \tilde{A}(x))$.

Obviously, $\tilde{A}_q$ is the characteristic function of $A_q = \{x \in A | \theta \in q * \tilde{A}(x)\}$, called a cut subset or a $q$-cut of $\tilde{A}$. Note that $A_\theta = A$.

Example 3.3. Let $A = \{x, y\}$ be a set and let $H = \{\theta, a, b\}$ be a hyper BCK-algebra with the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\theta$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>${\theta}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$a$</td>
<td>${\emptyset}$</td>
<td>${\theta, a}$</td>
<td>${a}$</td>
</tr>
<tr>
<td>$b$</td>
<td>${\emptyset}$</td>
<td>${\emptyset}$</td>
<td>${\emptyset, \theta, b}$</td>
</tr>
</tbody>
</table>

The mapping $\tilde{A} : A \to H$ given by $\tilde{A} = \begin{pmatrix} x & y \\ a & b \end{pmatrix}$ is a hyper BCK-function. Its cut subsets are $A_\theta = A$, $A_a = \{x\}$, $A_b = \{y\}$.

Proposition 3.4. Every hyper BCK-function $\tilde{A} : A \to H$ on $A$ is represented by the supremum of the set $\{q \in X | \tilde{A}_q(x) = 1\}$, that is $(\forall x \in A) (\tilde{A}(x) = \sup\{q \in X | \theta \in q * \tilde{A}(x)\})$.

Proof. For any $x \in A$, let $\tilde{A}(x) = r \in H$. Then $\theta \in r * \tilde{A}(x)$ and so $\tilde{A}_r(x) = 1$. 


Assume that \( \tilde{A}_p(x) = 1 \) for \( p \in H \), then \( \theta \in p * \tilde{A}(x) = p * r \). Thus \( p \ll r \).

Since \( r \in \{ p \in H | \tilde{A}_p(x) = 1 \} \), it follows that \( \tilde{A}(x) = r = \sup \{ p \in H | \tilde{A}_p(x) = 1 \} \). \( \square \)

For a hyper BCK-function \( \tilde{A} : A \to H \) on \( A \), consider the following sets:

\( A_H := \{ A_q | q \in H \}; \tilde{A}_H := \{ \tilde{A}_q | q \in H \} \).

**Proposition 3.5.** If \( \tilde{A} : A \to H \) is a hyper BCK-function on \( A \), we can easily obtain the following results:

1. \( \forall x \in A \) \( (\tilde{A}(x) = \sup \{ q * \tilde{A}_q(x) | q \in H \}) \),
2. where \( q * \tilde{A}_q(x) = \begin{cases} q & \text{if } \tilde{A}_q(x) = 1; \\ \theta & \text{otherwise}. \end{cases} \)
3. \( \forall q, p \in H \) \( (\theta \in p * q \iff A_q \subseteq A_p) \),
4. \( \forall x, y \in A \) \( (\tilde{A}(x) \neq \tilde{A}(y) \iff A_{\tilde{A}(x)} \neq A_{\tilde{A}(y)}) \),
5. \( \forall q \in H \) \( (\forall x \in A \) \( (q * \tilde{A}(x) = \{ \theta \} \iff A_{\tilde{A}(x)} \subseteq A_q) \),
6. \( \forall x, y \in A \) \( (\tilde{A}(x) * \tilde{A}(y) = \{ \theta \} \iff A_{\tilde{A}(y)} \subseteq A_{\tilde{A}(x)}) \),
7. \( \forall Y \subseteq H \) \( (\exists \sup Y \in H \Rightarrow A_{\sup \{ q | q \in Y \}} = \bigcap \{ A_q | q \in Y \}) \). (Note here that the sup is define via the hyperorder “ \( \ll \) ”),
8. \( \forall S \subseteq H \) \( (A_{\sup \{ q | q \in S \}} = \bigcap \{ A_q | q \in S \}) \),
9. \( \forall Y \subseteq H \) \( (\exists \sup Y \in H \Rightarrow A_{\sup \{ q | q \in Y \}} = \bigcap \{ A_q | q \in Y \}) \).

The following example shows that the converse of (viii) may not be true in general.

**Example 3.6.** Let \( A = \{ x, y \} \) be a set and let \( H = \{ \theta, a, b, c \} \) be a hyper BCK-algebra with the following table:

\[
\begin{array}{c|c|c|c|c}
* & \theta & a & b & c \\
\hline
\theta & \{ \theta \} & \{ \theta \} & \{ \theta \} & \{ \theta \} \\
a & \{ a \} & \{ \theta, a \} & \{ \theta \} & \{ a \} \\
b & \{ b \} & \{ a \} & \{ \theta \} & \{ b \} \\
c & \{ c \} & \{ c \} & \{ c \} & \{ \theta \} \\
\end{array}
\]

The function \( \tilde{A} : H \to A \) given by \( \tilde{A} = \begin{pmatrix} x & y \\ a & c \end{pmatrix} \) is a hyper BCK-function on \( A \) and the cut sets of \( \tilde{A} \) are as follows: \( A_\theta = A, A_a = \{ x \}, \)
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\[ A_b = \emptyset, \ A_c = \{y\}. \]

\[ \sup\{a, c\} \text{ does not exist but } A_a \cap A_c \in A_H. \]

## 4. Codes generated by hyper BCK-functions

Let \( \tilde{A} : A \to H \) be a hyper BCK-function on \( A \) and let \( \sim \) be a binary relation on \( H \) defined by \((\forall p, q \in H) (p \sim q \iff A_p = A_q)\). Then \( \sim \) is clearly an equivalence relation on \( H \).

Let \( \tilde{A}(A) := \{q \in H | \tilde{A}(x) = q \text{ for some } x \in A\} \).

Let \( x/\sim = \{y \in H | x \sim y\} \), for any \( x \in H \). \( x/\sim \) is called equivalence class containing \( x \). It is also easy to see that \( \tilde{A}(x) = \sup(x/\sim) \) is the greatest element of \( \sim \)-class to which it belongs and that every \( \sim \)-class contains exactly one element.

### 4.1. From a hyper BCK-algebra to a block code

Let \( A = \{1, 2, \ldots, n\} \) and let \( H \) be a finite hyper BCK-algebra. Every hyper BCK-function \( \tilde{A} : A \to H \) on \( A \) determines a binary block code \( V \) of length \( n \) in the following way:

To every \( x/\sim \), where \( x \in H \), there corresponds a codeword \( v_x = x_1x_2\ldots x_n \) such that \( x_i = j \iff \tilde{A}_c(i) = j \) for \( i \in A \) and \( j \in \{0,1\} \).

Let \( v_x = x_1x_2\ldots x_n \) and \( v_y = y_1y_2\ldots y_n \) be two codewords belonging to a binary block code \( V \). We can define an order relation \( \leq_c \) on the set codewords belonging to a binary block code \( V \) as follows:

\[ v_x \leq_c v_y \iff y_i \leq x_i \text{ for } i = 1, 2, \ldots, n. \]

**Example 4.1.** (1) Let \( H = \{0, 1, 2\} \) be a hyper BCK-algebra defined by the following table:

<table>
<thead>
<tr>
<th>* ( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0} &amp; {0} &amp; {0}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>{1} &amp; {0} &amp; {0}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>{2} &amp; {2} &amp; {0,2}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Let \( \tilde{A} : H \to H, x \mapsto \begin{cases} 0, & \text{if } x=0; \\ 1, & \text{if } x=1; \\ 2, & \text{if } x=2. \end{cases} \)

\[
\begin{array}{ccc}
A_0 & 0 & 1 & 2 \\
A_1 & 1 & 1 & 1 \\
A_2 & 0 & 1 & 0 \\
\end{array}
\]

Then \( V = \{111, 010, 001\} \) and
(2) Let $H = \{\theta, a, b, c\}$ be a hyper BCK-algebra defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\theta$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>${\theta}$</td>
<td>${\theta}$</td>
<td>${\theta}$</td>
<td>${\theta}$</td>
</tr>
<tr>
<td>$a$</td>
<td>${a}$</td>
<td>${\theta, a}$</td>
<td>${\theta, a}$</td>
<td>${\theta, a}$</td>
</tr>
<tr>
<td>$b$</td>
<td>${b}$</td>
<td>${b}$</td>
<td>${\theta, a}$</td>
<td>${\theta, a}$</td>
</tr>
<tr>
<td>$c$</td>
<td>${c}$</td>
<td>${c}$</td>
<td>${c}$</td>
<td>${\theta, a}$</td>
</tr>
</tbody>
</table>

Let $\tilde{A} : H \rightarrow H$ be a hyper BCK-function on $H$ given by $\tilde{A} = \begin{pmatrix} \theta & a & b & c \\ \theta & a & b & c \end{pmatrix}$

Then

\[
\begin{array}{c|cccc}
\theta & a & b & c \\
\hline
A_\theta & 1 & 1 & 1 & 1 \\
A_a & 0 & 1 & 1 & 1 \\
A_b & 0 & 0 & 1 & 1 \\
A_c & 0 & 0 & 0 & 1 \\
\end{array}
\]

Thus $V = \{1111, 0111, 0011, 0001\}$ and
Theorem 4.2. Every finite hyper BCK-algebra $H$ determines a block codes $V$ such that $(H, \ll)$ is isomorphic to $(V, \leq_c)$.

Proof. Let $H = \{a_1, a_2, ..., a_n\}$ be a finite hyper BCK-algebra in which $a_1 = \theta$ and let $A : H \rightarrow H$ be the identity hyper BCK-function on $H$. The decomposition of $A$ provides a family $\{\bar{A}_q|q \in H\}$ which is the desired code under the order $(\leq_c)$. Let $f : H \rightarrow \{\bar{A}_q|q \in H\}$ be a function defined by $f(q) = \bar{A}_q$ for all $q \in H$. Since every $\sim$-class contains exactly one element, hence $f$ is one-to-one.

Let $x \in H$ and $p, q \in H$ be such that $p \ll q$. If $\bar{A}_q(x) = 0$, then $\bar{A}_p(x) \leq \bar{A}_p(x)$. If $\bar{A}_q(x) = 1$, then $\theta \in q \ast \bar{A}(x)$, i.e. $q \ll \bar{A}(x)$. Thus $P \ll q$ and $q \ll \bar{A}(x)$ by using the transitivity of the relation $\ll$, we obtain $p \ll \bar{A}(x)$, i.e. $\theta \in p \ast \bar{A}(x)$. Therefore $\bar{A}_p(x) = 1$ and we conclude that $\bar{A}_p \leq_c \bar{A}_q$.

Therefore $f$ is an isomorphism. \hfill \Box

4.2. From a binary block code to a hyper BCK-functions.

Example 4.3. Let $(H, \leq)$ be a finite partial ordered set with the minimum element denoted by $\theta$. We define the following hyper operation $\ast$ on $H$:

\[
\begin{aligned}
\{ \theta \ast x = \{\theta\} \text{ and } x \ast x = \{\theta\} \}, & \quad x \in H; \\
x \ast y = \{\theta\}, \text{ if } x \leq y & \quad x, y \in H; \\
x \ast y = \{x\}, \text{ if } y < x & \quad x, y \in H; \\
x \ast y = \{y\}, \text{ if } x, y \text{ can't be compared} & \quad x, y \in H.
\end{aligned}
\]

It is easy to see that $(H, \ast, \theta)$ is a hyper BCK-algebra.

If the above example of hyper BCK-algebra has $n$ elements, we will denote it with $C_n$. Let $V$ be a binary block code with $n$ codewords of length $n$. We consider the matrix $M_V = (m_{i,j})_{i,j \in \{1,2,...,n\}} \in M_n(\{0,1\})$ with the rows consisting of the codewords of $V$. This matrix is called the matrix associated to the code $V$.

Theorem 4.4. With the above notations, if the codeword $\overline{11...1}$ is in $V$ and the matrix $M_V$ is upper triangular with $m_{ii} = 1$, for all $i \in \{1,2,...,n\}$, there are a set $A$ with $n$ elements, a hyper BCK-algebra $H$ and a hyper BCK-function $f : A \rightarrow H$ such that $f$ determines $V$. 

\end{document}
Proof. We consider on $V$ the lexicographic order, denoted by $\leq_{\text{lex}}$. It is clear that $(V, \leq_{\text{lex}})$ is a totally ordered set.

Let $V = \{w_1, w_2, \ldots, w_n\}$, with $w_1 \geq_{\text{lex}} w_2 \geq_{\text{lex}} \ldots \geq_{\text{lex}} w_n$. This implies that $w_1 = 11 \cdots 1$ and $w_n = 00 \cdots 0 1$. On $V$, we define a partial order $\leq_{c}$ as in construction of the code by the hyper BCK-function. Now, $(V, \leq_{c})$ is a partially ordered set with $w_1 \leq_{c} w_i \leq_{c} w_n$, $i \in \{2, \ldots, n - 1\}$.

We remark that $w_1$ correspond to $\theta$ and $w_n$ is the maximal element in $(V, \leq_{c})$.

We define on $(V, \leq_{c}, \theta)$ a hyper operation $" \ast "$ as in Example 4.3.

Then $H = (V, \ast, \theta)$ is a hyper BCK-algebra and $V$ is isomorphic to $H$.

We consider $A = V$ and the identity map $f : A \to H$, $w \mapsto w$, as a hyper BCK-function on $A$. The decomposition of $f$ provides a family $V_H = \{f_r : A \to \{0, 1\} | f_r(x) = 1 \iff \theta \leq x \ast f(x), \forall x \in A, r \in H\}$.

This family is the binary block-code $V$ relative to the order relation $\leq_{c}$. Indeed, let $w_k \in V$, $1 < k < n$, then $w_k = 00 \cdots 0 x_{i_k} \cdots x_{i_n}$, with $x_{i_k}, \ldots, x_{i_n} \in \{0, 1\}$.

$\forall j \in (k-1)-\text{time}$ If $x_{i_j} = 0$, it result that $w_k \leq_{c} w_{i_j}$ and $\theta \leq w_k \ast w_{i_j}$.

If $x_{i_j} = 1$, we obtain that $w_{i_j} \leq_{c} w_k$ or $w_{i_j} \ast w_k$ can’t be compared, therefore $w_k \ast w_{i_j} = \{w_k\}$ or $w_k \ast w_{i_j} = \{w_{i_j}\}$. $\square$

The following example show that a binary block code as in Theorem 4.1 can be determined by two or more hyper BCK-algebras.

Example 4.5. Let $V = \{000010, 000110, 011101, 111111, 001011, 000001\}$ be a binary block code. Using the lexicographic order, the code $V$ can be written $V = \{111111, 011101, 001011, 000010, 000001\} = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. Define the partial order $\leq_{c}$ on $V$, we remark that $w_1 \leq_{c} w_i$ for $i \in \{2, 3, 4, 5, 6\}$; $w_2 \leq_{c} w_6$; $w_3 \leq_{c} w_5, w_6$; $w_4 \leq_{c} w_5$; $w_2$ can’t be compared with $w_3, w_4, w_5$; $w_4$ cant be compared with $w_3, w_6$; and $w_5$ cant be compared with $w_6$. The operation $" \ast "$ on $V$ is given in the following table:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>$w_5$</th>
<th>$w_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
</tr>
<tr>
<td>$w_2$</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
</tr>
<tr>
<td>$w_3$</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
</tr>
<tr>
<td>$w_4$</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
</tr>
<tr>
<td>$w_5$</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
</tr>
<tr>
<td>$w_6$</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
<td>{}</td>
</tr>
</tbody>
</table>
Obviously, $V$ with the operation " $\ast$ " is a hyper BCK-algebra. We remark that the same binary block code $V$ can be obtained from the hyper BCK-algebra $(H, \ast, \theta)$ with hyper BCK-function $\tilde{A} : V \to V$, $\tilde{A}(x) = x$. From the associated Cayley multiplication tables, it is obvious that the hyper BCK-algebras $(H, \ast, \theta)$ and $(V, \ast, w_1)$ are not isomorphic. From here, we obtain that hyper BCK-algebra associated to a binary block-code as in Theorem 4.4 is not unique up to an isomorphism.

**Proposition 4.6.** With the above notations, we consider $V$ as a binary block code with $n$ codewords of length $m$ (with $n \neq m$), or a block-code with $n$ codewords of length $n$ such that the codeword $11\ldots1_{\text{n-time}}$ is not in $V$, or a block-code with $n$ codewords of length $n$ such that the matrix $M_V$ is not upper triangular. There are a natural number $q = \max\{m, n\}$, a set $A$ with $m$ elements and a hyper BCK-function $\tilde{A} : A \to C_q$, where $C_q$ denote the hyper BCK-algebra with $q$ elements, such that the obtained block-code $V_{C_q}$ contains the block-code $V$ as a subset.

**Proof.** Let $C$ be a binary block-code, $C = \{w_1, w_2, \ldots, w_n\}$, with codewords of length $m$. We consider the codewords $w_1, w_2, \ldots, w_n$ lexicographic ordered, $w_1 \leq_{\text{lex}} w_2 \leq_{\text{lex}} \ldots \leq_{\text{lex}} w_n$. Let $M \in \mathcal{M}_{n,m}(\{0,1\})$ be the associated matrix with the rows $w_1, w_2, \ldots, w_n$ in this order. Using Proposition 2.8 in [3], we can extend the matrix $M$ to a square matrix $M' \in \mathcal{M}_p(\{0,1\})$, $p = m + n$; such that $M' = (m'_{ij})_{i,j \in \{1,2,\ldots,p\}}$ is an upper triangular matrix with $m'_{ii} = 1$, for all $i \in \{1,2,\ldots,p\}$. If the first line of the matrix $M'$ is not $11\ldots1_{\text{p-time}}$, then we insert the row $11\ldots1_{\text{p-time}}$ as a first row and the $10\ldots0_{\text{p+1-time}}$ as a first column. Let $q = p + 1$, applying Theorem 4.4 for the matrix $M'$, we obtain a residuated lattice $C_q = \{x_1, x_2, \ldots, x_q\}$, with $x_1$ correspond to 0 and $x_q$ correspond to 1, and a binary block-code $V_{C_q}$. Assuming that the initial column of the matrix
M have in the new matrix \( M' \) positions \( i_{j_i}, i_{j_2}, ..., i_{j_n} \in \{1, 2, ..., q\} \), let 
\[ A = \{ x_{j_1}, x_{j_2}, ..., x_{j_n} \} \subset Cq. \]
The hyper BCK-function \( f : X \rightarrow Cq \)
is such that \( f(x_{j_i}) = x_{j_i}, i \in \{1, 2, ..., m\} \), determines the binary-block code \( C_q \) such that \( C \subseteq V_{C_q} \) as restriction of the hyper BCK-function 
\[ f : C_q \rightarrow C_q \]
on \( A \) such that \( f(x_i) = x_i. \)

Let \( C \) be a binary block code with \( m \) codewords of length \( q \), with the above notations, let \( H \) be the associated BCK-algebra and \( W = \{\theta, w_1, \ldots, w_{m+q}\} \) the associated binary block code which include the code \( C \). We consider the codewords \( \theta, w_1, \ldots, w_{m+q} \) lexicographic ordered, \( \theta \geq_{\text{lex}} w_1 \geq_{\text{lex}} w_2 \geq_{\text{lex}} \ldots \geq_{\text{lex}} w_{m+q} \). Let \( M \in M_{m+q+1}(\{0, 1\}) \) be the associated matrix with the rows \( \theta, w_1, \ldots, w_{m+q} \) in this order. We denote with \( L_{w_i} \) and \( C_{w_j} \) the lines and columns in the matrix \( M \). The submatrix \( M' \) of the matrix \( M \) with the rows \( L_{w_1}, \ldots, L_{w_m} \) and the columns \( C_{w_{m+1}}, \ldots, C_{w_{m+q}} \) is the matrix associated to the code \( C \).

**Remark 4.7.** 1) If there exists \( x \in \{w_1, w_2, \ldots, w_m\} \) and \( y \in \{\theta, w_{m+1}, \ldots, w_{m+q}\} \) such that \( x \ll y \), then the set \( I = \{\theta, w_{m+1}, \ldots, w_{m+q}\} \) can’t be hyper BCK-ideal.

2) On \( W \), due to the order \( \leq_c \) given in the construction of code from a hyper BCK-function and to the hyper operation \( \ast \) define in Example 4.3, for the product of two elements of \( W \), we can have only two possibilities \( w_i * w_j = \{\theta\} \) or \( w_i * w_j = \{w_i\}, (w_i, w_j \in W \) and \( i, j \in \{1, m + q\} \).

**Example 4.8.** Let \( V = \{101, 110\} \) be a binary block code. Using the lexicographic order, the code \( V \) can be written \( V = \{110, 101\} = \{w_1, w_2\}. \)

Let \( M_V = \left( \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right) \) be the associated matrix. By Proposition 2.8 in [3],

we construct the matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5 \\
w_6
\end{pmatrix}.
\]

The binary block code \( W = \{w_1, w_2, w_3, w_4, w_5, w_6\} \), determines a hyper BCK-algebra \( (H, *, w_1) \). Let \( X = \{w_4, w_5, w_6\} \) and \( A : X \rightarrow W \), \( w_i \mapsto w_i, (i \in \{4, 5, 6\}) \) be a hyper BCK-function which determines the binary block code \( U = \{111, 110, 101, 100, 010, 001\} \). Remark that the code \( V \) is a subset of the code \( U \).
Proposition 4.9. Out of the above notations, if we assume that there is not \( x \in \{ w_1, w_2, ..., w_n \} \) such that for any \( y \in I; x \ll y \). Then, \( I \) determines a hyper BCK-ideal in the hyper BCK-algebra \( H \).

Proof. since \( \theta \in I \), it will be sufficient to prove the property \((H1_2)\) for these hyper BCK-ideal.

Let \( x, y \in H \) such that \( x \ast y \ll I \) and \( y \in I \).

If \( x, y \) are not compared or \( x \gg y \) and if \( x \notin I \), then \( x \ast y = \{ x \} \ll I \).

Since \( x \notin I \), then \( \theta \in x \ast z = \{ x \} \). Thus \( x = \theta \in I \) contradiction. Therefore \( x \in I \).

If \( x \ll y \), with \( y \in I \), then \( x \in I \). \( \Box \)

Example 4.10. Let \( V = \{000\} = \{w_1\} \) be a binary block code. Let \( M_V = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \) be the associated matrix. By Proposition 2.8 in [3], we construct the matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5
\end{pmatrix}
\]

The binary block code \( W = \{ w_1, w_2, w_3, w_4, w_5 \} \), determines a hyper BCK-algebra \((H, \ast, w_1)\). Let \( X = \{ w_3, w_4, w_5 \} \) and \( \tilde{A} : X \rightarrow W, w_i \mapsto w_i, (i \in \{3, 4, 5\}) \) be a hyper BCK-function which determines the binary block code \( U = \{111, 000, 1000, 010, 001\} \). It is clear that the code \( V \) is a subset of the code \( U \).

Since there are not \( w_2 \leq_c w_3, w_4 \) and \( w_5 \), then set \( I = \{ w_1, w_4, w_5, w_6 \} \) is a hyper BCK-ideal.

5. CONCLUSION

In this work, we have studied the connection between hyper BCK-algebras and coding theory. We have also proved that to each hyper BCK-algebras (hyper BCK-function) we can associated a binary block codes. Moreover we establish the link between the hyper BCK-algebra constructed from a binary block code and hyper BCK-ideal on a hyper BCK-algebra.

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